[東京工業大学 1988 年 5]

$$\lim_{n\to\infty}\left(\frac{{}_{3n}\mathrm{C}_n}{{}_{2n}\mathrm{C}_n}\right)^{\frac{1}{n}}$$ を求めよ。

${}_{3n}\mathrm{C}_n=\dfrac{(3n)!}{n!(2n)!}$，${}_{2n}\mathrm{C}_n=\dfrac{(2n)!}{n!n!}$ であるから

$$\frac{{}_{3n}\mathrm{C}_n}{{}_{2n}\mathrm{C}_n}=\frac{\dfrac{(3n)!}{n!(2n)!}}{\dfrac{(2n)!}{n!n!}}$$

$$=\frac{3n(3n-1)\cdots\cdots(2n+1)}{2n(2n-1)\cdots\cdots(n+1)}$$

よって

$$\log\left(\frac{{}_{3n}\mathrm{C}_n}{{}_{2n}\mathrm{C}_n}\right)^{\frac{1}{n}}=\frac{1}{n}\log\frac{{}_{3n}\mathrm{C}_n}{{}_{2n}\mathrm{C}_n}$$

$$=\frac{1}{n}\log\frac{3n(3n-1)\cdots\cdots(2n+1)}{2n(2n-1)\cdots\cdots(n+1)}$$

$$=\frac{1}{n}\left(\log\frac{3}{2}+\log\frac{3n-1}{2n-1}+\cdots+\log\frac{2n+1}{n+1}\right)$$

$$=\frac{1}{n}\sum_{k=0}^{n-1}\log\frac{3n-k}{2n-k}$$

$$=\frac{1}{n}\sum_{k=0}^{n-1}\log\frac{3-\dfrac{k}{n}}{2-\dfrac{k}{n}}$$

したがって

$$\lim_{n\to\infty}\log\left(\frac{{}_{3n}\mathrm{C}_n}{{}_{2n}\mathrm{C}_n}\right)^{\frac{1}{n}}=\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\log\frac{3-\dfrac{k}{n}}{2-\dfrac{k}{n}}$$

$$=\int_0^1\log\frac{3-x}{2-x}\,dx$$

$$=\int_0^1\log(3-x)\,dx-\int_0^1\log(2-x)\,dx$$

$$= -2(\log 2 - 1) + 3(\log 3 - 1) - \left\{+1 + 2(\log 2 - 1)\right\}$$

$$= \left[-(3-x)\left\{\log\left(3-x\right)-1\right\}\right]_0^1 - \left[-(2-x)\left\{\log\left(2-x\right)-1\right\}\right]_0^1$$

$$= -4\log 2 + 3\log 3$$

$$= \log\frac{27}{16}$$

よって $\displaystyle\lim_{n\to\infty}\left(\frac{{}_{3n}\mathrm{C}_n}{{}_{2n}\mathrm{C}_n}\right)^{\frac{1}{n}} = \frac{27}{16}$