１１９．いろいろな数列の和

(1) $\dfrac{n}{3n+1}$　(2) $\sqrt{n+3}-\sqrt{3}$　(3) $\dfrac{n^2+2n}{(n+1)^2}$　(4) $\dfrac{(4n-1)\cdot 5^n+1}{16}$

次の和を求めよ。

(1) $$\sum_{k=1}^{n}\frac{1}{(3k-2)(3k+1)}=\sum_{k=1}^{n}\frac{1}{3}\left(\frac{1}{3k-2}-\frac{1}{3k+1}\right)$$

$$=\frac{1}{3}\left\{\left(\frac{1}{1}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{7}\right)+\cdots\cdots+\left(\frac{1}{3n-2}-\frac{1}{3n+1}\right)\right\}$$

$$=\frac{1}{3}\left(1-\frac{1}{3n+1}\right)$$

$$=\frac{n}{3n+1}$$

(2) $$\sum_{k=1}^{n}\frac{1}{\sqrt{k+2}+\sqrt{k+3}}=\sum_{k=1}^{n}\frac{1}{\sqrt{k+2}+\sqrt{k+3}}\cdot\frac{\sqrt{k+2}-\sqrt{k+3}}{\sqrt{k+2}-\sqrt{k+3}}$$

$$=\sum_{k=1}^{n}\frac{\sqrt{k+2}-\sqrt{k+3}}{(k+2)-(k+3)}$$

$$=-\sum_{k=1}^{n}\left(\sqrt{k+2}-\sqrt{k+3}\right)$$

$$=-\left\{\left(\sqrt{3}-\sqrt{4}\right)+\left(\sqrt{4}-\sqrt{5}\right)+\cdots\cdots+\left(\sqrt{n+2}-\sqrt{n+3}\right)\right\}$$

$$=\sqrt{n+3}-\sqrt{3}$$

(3) $$\sum_{k=1}^{n}\frac{2k+1}{k^2(k+1)^2}=\sum_{k=1}^{n}\left\{\frac{1}{k^2}-\frac{1}{(k+1)^2}\right\}$$

$$=\left\{\left(\frac{1}{1^2}-\frac{1}{2^2}\right)+\left(\frac{1}{2^2}-\frac{1}{3^2}\right)+\cdots\cdots+\left(\frac{1}{n^2}-\frac{1}{(n+1)^2}\right)\right\}$$

$$=1-\frac{1}{(n+1)^2}$$

$$=\frac{n^2+2n}{(n+1)^2}$$

（4）$\displaystyle\sum_{k=1}^{n} k\cdot 5^{k-1}$

$S_n=\displaystyle\sum_{k=1}^{n} k\cdot 5^{k-1}$ とおく。

$$S_n = 1\cdot 5^0+2\cdot 5^1+3\cdot 5^2+\cdots\cdots\cdots\cdots\cdots+n\cdot 5^{n-1} \quad \cdots①$$

$$5S_n = \qquad 1\cdot 5^1+2\cdot 5^2+3\cdot 5^3+\cdots\cdots+(n-1)\cdot 5^{n-1}+n\cdot 5^n \quad \cdots②$$

①－②より

$$-4S_n = 1+5^1+5^2+5^3+\cdots\cdots+5^{n-1}-n\cdot 5^n$$

$$=\frac{(1-5^n)}{1-5}-n\cdot 5^n$$

$$=-\frac{(4n-1)\cdot 5^n+1}{4}$$

したがって　$S_n=\dfrac{(4n-1)\cdot 5^n+1}{16}$

〔別解〕

$T_n=\displaystyle\sum_{k=1}^{n} x^k$　とおく。

$T_n=1+x+x^2+x^3+\cdots\cdots+x^n=\dfrac{1-x^{n+1}}{1-x}$　であり，この式の両辺を x で微分すると

$$1+2x+3x^2+\cdots\cdots+nx^{n-1}=\frac{-(n+1)x^n\cdot(1-x)-(1-x^{n+1})\cdot(-1)}{(1-x)^2}$$

$$=\frac{(n+1)(x^{n+1}-x^n)+1-x^{n+1}}{(1-x)^2}$$

$$=\frac{nx^{n+1}-(n+1)x^n+1}{(1-x)^2} \quad \cdots（＊）$$

（＊）において，$x=5$ を代入した式が $\displaystyle\sum_{k=1}^{n} k\cdot 5^{k-1}$　であるから

$$1+2\cdot 5+3\cdot 5^2+\cdots\cdots+n\cdot 5^{n-1}=\frac{n\cdot 5^{n+1}-(n+1)\cdot 5^n+1}{(1-5)^2}$$

$$=\frac{5n\cdot 5^n-n\cdot 5^n-5^n+1}{16}$$

$$=\frac{(4n-1)\cdot 5^n+1}{16}$$